disc 104

Wednesday, March 3, 2021

Definition 3.10 Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a **cubic spline interpolant** S for f is a function that satisfies the following conditions:

has no conditions direction at its curve takes the it line after it e interpolation endpoints. The m the fact that l shape a flexible forced to pass l interpolation lditional Figure 3.9.)

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$;

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$;

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$; (*Implied by* **(b)**.)

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(f) One of the following sets of boundary conditions is satisfied:

(i) $S''(x_0) = S''(x_n) = 0$ (**natural** *(or free)* **boundary**);

(ii) $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped boundary**).

3.5 Problems

Problem 1. *Determine the natural cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 1$, $f(2) = 2$.*

Problem 2. *Determine the clamped cubic spline s that interpolates the data $f(0) = 0, f(1) = 1, f(2) = 2$ and satisfies $s'(0) = s'(2) = \cancel{0}$ 1*

Problem 3. *Suppose $\{x_i f(x_i)\}_{i=1}^{n}$ lie on a straight line. What can be said about the natural and clamped cubic splines for the function f?*

1) need to find cubic polynomial $P_1(x)$ on $[0,1]$

& - - - $P_2(x)$ on $[1,2]$

$P_1(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$P_2(x) = b_0 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3$

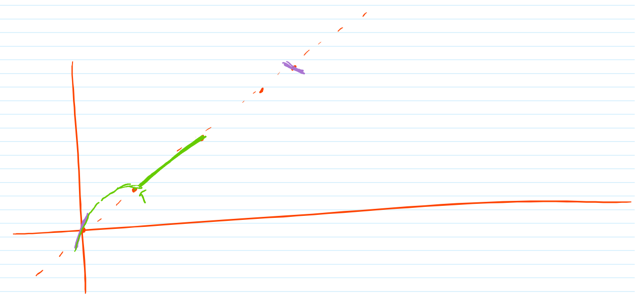
$P_1(0) = 0$
$P_1(1) = 1$
$P_2(1) = 1$
$P_2(2) = 2$

$P_1'(1) = P_2'(1)$
$P_1''(1) = P_2''(1)$
$\cancel{P_1''(0) = 0}$
$\cancel{P_2''(2) = 0}$

$\Rightarrow$

$a_0 = 0$
$b_0 = 1$
$a_1 + a_2 + a_3 = 1$
$1 + b_1 + b_2 + b_3 = 2$
$\cancel{2a_2 = 0}$
$\cancel{2b_2 + 12b_3 = 0}$
$a_1 + 2a_2 + 3a_3 = b_1$
$2a_2 + 6a_3 = 2b_2$

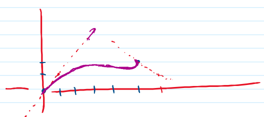
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 12 \\ 1 & 2 & 3 & -1 & 0 & 0 \\ 0 & 2 & 6 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



2) $P_1'(0) = 1$
$P_2'(2) = 1$ $\Rightarrow$

$a_1 = 1$
$b_1 + 2b_2 + 3b_3 = 1$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & -1 & 0 & 0 \\ 0 & 2 & 6 & 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$



3.6 Problems

Problem 4. *Let $(x_0, y_0) = (0,0)$ and $(x_1, y_1) = (5,2)$ be the endpoints of a curve. Use the given guidepoints to construct parametric cubic Hermite approximations $(x(t), y(t))$ to the curve and graph the approximations (a) $(1,1)$ and $(6,1)$ ~~(b) $(2,2)$ and $(7,0)$~~*

polynomial $x(t)$ on $[0, 1]$ satisfies

$$x(0) = x_0, \quad x(1) = x_1, \quad x'(0) = \alpha_0, \quad \text{and} \quad x'(1) = \alpha_1.$$

The unique cubic polynomial satisfying these conditions is

$$x(t) = [2(x_0 - x_1) + (\alpha_0 + \alpha_1)]t^3 + [3(x_1 - x_0) - (\alpha_1 + 2\alpha_0)]t^2 + \alpha_0 t + x_0. \quad (3.23)$$

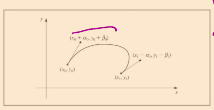
In a similar manner, the unique cubic polynomial satisfying

$$y(0) = y_0, \quad y(1) = y_1, \quad y'(0) = \beta_0, \quad \text{and} \quad y'(1) = \beta_1$$

is

$$y(t) = [2(y_0 - y_1) + (\beta_0 + \beta_1)]t^3 + [3(y_1 - y_0) - (\beta_1 + 2\beta_0)]t^2 + \beta_0 t + y_0. \quad (3.24)$$

Figure 3.17



$(x(t), y(t))$ $t \in [0,1]$

want cubic $x(t)$ s.t.

$x(0) = 0$, $x(1) = 5$, $x'(0) = 1$, $x'(1) = -1$

$(1,1) = (x_0 + \alpha_0, y_0 + \beta_0)$
$= (0 + \alpha_0, 0 + \beta_0)$

$\alpha_0 = 1$
$\beta_0 = 1$

$(6,1) = (x_1 - \alpha_1, y_1 - \beta_1)$
$= (5 - \alpha_1, 2 - \beta_1)$

$\Rightarrow$ $\alpha_1 = -1$
$\beta_1 = 1$

$$\boxed{\begin{aligned} x(t) &= t + 14t^2 - 10t^3 \\ y(t) &= t + 3t^2 - 2t^3 \end{aligned}}$$

4.1 Problems

Problem 5. *Use the forward-difference formulas and backward-difference formulas to determine each missing entry in the following table:*

x	$f(x)$	$f'(x)$
.5	.4794	
.6	.5646	
.7	.6442	

Problem 6. *For the previous problem, $f(x) = \sin(x)$. Determine the actual error and find error bounds using the error formulas.*

$D_hf(x_0)$ error.

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi). \quad (4.1)$$

For small values of h, the difference quotient $[f(x_0 + h) - f(x_0)]/h$ can be used to approximate $f'(x_0)$ with an error bounded by $M|h|/2$, where M is a bound on $|f''(x)|$ for x between x_0 and $x_0 + h$. This formula is known as the **forward-difference formula** if $h > 0$ (see Figure 4.1) and the **backward-difference formula** if $h < 0$.

Bound $|f'(x_0) - D_hf(x_0)| = \frac{h}{2}|f''(\xi)|$

$= \frac{h}{2}|\sin(\xi)| \le \frac{.1}{2} \cdot 1 = \frac{1}{20} = .05$

$h = .1$